

Variations of selective separability and tightness in function spaces with set open topologies

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Abstract

We study tightness properties and selective versions of separability in bitopological function spaces endowed with set-open topologies.

Keywords:

Selection principles, bitopology, selective separability, set-open topology, C -compactness, submetrizable, fan tightness, strong fan tightness, T -tightness, R -separability, M -separability, GN -separability, H -separability

2000 MSC: 54C35, 54C40, 54D65, 46E55

1. Introduction

In this paper we are mainly concerned with selective versions of separability in bitopological function spaces endowed with two homogenous set-open topologies.

Variations of separability, stronger forms, weaker forms, functional separability and similar properties have been intensively studied by many mathematicians in the last several decades. The selection versions of separability has recently gained a particular attention and as a consequences many interesting results were obtained.

[☆]This work was supported by Act 211 Government of the Russian Federation, contract 02.A03.21.0006.

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Although the definition of the selection principles was given by Scheepers in 1996 the initial studies of the theory was based on the papers by Menger, Hurewicz, Rothberger and Sierpinski in 1920-1930, see [11, 16, 24].

Many topological properties can be defined or characterized in terms of the following two classical selection principles given in a general form in [26] as follows:

Let \mathcal{A} and \mathcal{B} be sets consisting of families of subsets of an infinite set X . Then:

$S_1(\mathcal{A}, \mathcal{B})$: for each sequence $(A_n : n \in \mathbb{N})$ of elements of \mathcal{A} there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n , $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of \mathcal{B} .

$S_{fin}(\mathcal{A}, \mathcal{B})$: for each sequence $(A_n : n \in \mathbb{N})$ of elements of \mathcal{A} there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n , $B_n \subseteq A_n$, and $\bigcup_{n \in \mathbb{N}} B_n \in \mathcal{B}$.

These selection principles denoted by $S_{fin}(\mathcal{O}, \mathcal{O})$ and $S_1(\mathcal{O}, \mathcal{O})$ are called Menger and Rothberger property where \mathcal{O} is the family of open covers of a topological space.

For the topological space X , let \mathcal{D} denote the family of dense subspaces of X . The selection principles $S_{fin}(\mathcal{D}, \mathcal{D})$ and $S_1(\mathcal{D}, \mathcal{D})$ were introduced by Scheepers in [27] and recently gained a great attention, see [3, 4, 5, 10].

In [3] the selection properties $S_{fin}(\mathcal{D}, \mathcal{D})$, $S_1(\mathcal{D}, \mathcal{D})$ and $S_1(\mathcal{D}, \mathcal{D}^{gp})$ are called M -separability (also called selective separability), R -separability and GN -separability, respectively, while a bit modified property $S_{fin}(\mathcal{D}, \mathcal{D}^{gp})$ is called H -separability where "M-", "R-" and "H-" represent well known Menger, Rothberger and Hurewicz properties.

It should be noted that very recently Tsaban and his co-authors in [6] studied all properties $S(\mathcal{A}, \mathcal{B})$ for $S \in \{S_1, S_{fin}\}$ and \mathcal{A}, \mathcal{B} are combinations of open covers, dense open families and dense sets.

The selection principle theory is firstly considered in bitopological spaces by Kočinac and Özçağ in [14, 15] and they carried out a systematic study on selection principles mainly selective versions of separability in bitopological spaces, particularly in the space $C(X)$ of all continuous real-valued functions defined on a Tychonoff space X , where $C(X)$ is endowed with the topology τ_p of pointwise topology and the compact-open topology τ_k .

In the following we investigate some properties of bitopological selective versions of separability in function spaces and the set-open topologies will be used as a main tool.

The set open topology on a family λ of nonempty subsets of the set X is a generalization of the compact open topology and of the topology of pointwise convergence. This notion was first introduced by Arens and Dugundji in [1] and was widely investigated by Osipov in [19, 20, 21]. In the next section we recall some facts on the set open topologies.

For the background material on selection principles we refer to the survey papers [13, 28, 29], for the undefined notions in function spaces, see [2]. We will follow [8] for topological terminology and notations.

2. Main definitions and notation

Recall that a subset A of a space X is called *C-compact subset of X* if, for any real-valued function f continuous on X , the set $f(A)$ is compact in \mathbb{R} .

Let X be a topological space. Then:

Ψ denotes the collection of all π -networks of closed *C-compact* subsets of the set X such that it is closed under *C-compact* subsets of the set X of its elements.

A family λ of *C-compact* subsets of X is said to be closed under (hereditary with respect to) *C-compact* subsets if it satisfies the following condition: whenever $A \in \lambda$ and B is a *C-compact* (in X) subset of A , then $B \in \lambda$ also.

Note that $p \in \Psi$ and $k \in \Psi$ where p and k are sets all finite and compact subsets of X .

We use the following notation for various topological spaces with the underlying set $C(X)$:

$C_\lambda(X)$ for the λ -open topology.

The element of the standard subbase of the λ -open topology:

$[F, U] = \{f \in C(X) : f(F) \subseteq U\}$ where $F \in \lambda$ and U is a open subset of \mathbb{R} .

Given a family λ of non-empty subsets of X , let $\lambda(C) = \{A \in \lambda : \text{for every } C\text{-compact subset } B \text{ of the space } X \text{ with } B \subset A, \text{ the set } [B, U] \text{ is open in } C_\lambda(X) \text{ for any open set } U \text{ of the space } \mathbb{R}\}$.

Let λ_m denote the maximal with respect to inclusion family, provided that $C_{\lambda_m}(X) = C_\lambda(X)$. Note that a family λ_m is unique for each family λ .

Interest in studying the λ -open topology generated by a Theorem 3.3 in [19] which characterizes some topological-algebraic properties of the set-open topology.

The following theorem is a corollary of Theorem 3.3 in [19].

Theorem 2.1. *For a space X , the following statements are equivalent.*

1. $C_\lambda(X)$ is a paratopological group.
2. $C_\lambda(X)$ is a topological group.
3. $C_\lambda(X)$ is a topological vector space.
4. $C_\lambda(X)$ is a locally convex topological vector space.
5. $C_\lambda(X)$ is a topological ring.
6. $C_\lambda(X)$ is a topological algebra.
7. λ is a family of C -compact sets and $\lambda = \lambda(C)$.
8. λ_m is a family of C -compact sets and it is hereditary with respect to C -compact subsets.

So without loss of generality we can assume that $C_\lambda(X)$ is a paratopological group (TVS, locally convex TVS) under the usual operations of addition and multiplication (and multiplication by scalars) iff $\lambda_m \in \Psi$.

Further, throughout the article, we assume $\lambda = \lambda_m \in \Psi$.

In particular, if $\lambda = p$ ($\lambda = k$) then $C_\lambda(X) = C_p(X)$ ($C_\lambda(X) = C_k(X)$), i.e. λ -open topology coincide with the topology of pointwise convergence (the compact-open topology).

Since $C_\lambda(X)$ is homogenous space we may always consider the point $\mathbf{0}$ when studying local properties of this space.

We use the symbol Ω_x to denote the set $\{A \subset X : x \in Cl_\tau(A) \setminus A\}$, where (X, τ) is a topological space and $x \in X$.

Since we are considering two topologies of $C(X)$ we shall use the symbol $(\Omega_0)^\lambda$ to denote Ω_0 in the space $C_\lambda(X)$ and the symbol $(\Omega_0)^\mu$ to denote Ω_0 in the space $C_\mu(X)$ where $\lambda, \mu \in \Psi$.

We will denote by τ_λ the λ -open topology on $C(X)$ for $\lambda \in \Psi$.

Before we proceed let us recall some definitions, notations and terminology which were used in [14, 15].

Throughout this paper (X, τ_1, τ_2) will be a bitopological space (shortly bispace), i.e. the set X endowed with two topologies τ_1 and τ_2 . For a subset A of X , Cl_i will denote the closure of A in (X, τ_i) , $i = 1, 2$.

A subset A of X is bidense (shortly d -dense) in (X, τ_1, τ_2) if A is dense in both (X, τ_1) and (X, τ_2) . X is d -separable if there is a countable set A which is d -dense in (X, τ_1, τ_2) .

Denote by \mathcal{D}_1 and \mathcal{D}_2 the collections of all dense subsets of (X, τ_1) and (X, τ_2) , respectively. We say that X is:

M_{τ_i, τ_j} -separable ($i, j = 1, 2; i \neq j$), if for each sequence $(D_n : n \in \mathbb{N})$ of elements of \mathcal{D}_i there are finite sets $F_n \subset D_n$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} F_n \in \mathcal{D}_j$, i.e. if $S_{fin}(\mathcal{D}_i, \mathcal{D}_j)$ hold;

R_{τ_i, τ_j} -separable if $S_1(\mathcal{D}_i, \mathcal{D}_j)$ hold;

H_{τ_i, τ_j} -separable if for each sequence $(D_n : n \in \mathbb{N})$ of elements of \mathcal{D}_i there are finite sets $F_n \subset D_n$, $n \in \mathbb{N}$, such that each τ_j -open subset of X intersects F_n for all but finitely many n ;

GN_{τ_i, τ_j} -separable if $S_1(\mathcal{D}_i, \mathcal{D}_j^{gp})$ hold.

Here \mathcal{D}_j^{gp} is the collection of groupable dense subsets of a space; a countable dense subset D of a space Z is groupable if $D = \bigcup_{n \in \mathbb{N}} A_n$, each A_n finite and each open set U in Z intersects all but finitely many A_n [9].

In case $\tau_1 = \tau_2 = \tau$, then these definitions coincide with definitions of corresponding topological selective versions of separability of (X, τ) .

As mentioned in [14] we have the implications, GN_{τ_i, τ_j} -separable $\implies R_{\tau_i, \tau_j}$ -separable $\implies M_{\tau_i, \tau_j}$ -separable, and H_{τ_i, τ_j} -separable $\implies M_{\tau_i, \tau_j}$ -separable.

The remaining notations can be found in [8, 14, 15].

3. The tightness-type properties

In this section we give some results on bitopological versions of the tightness properties and its variations in function bispaces. We also combine these results with bitopological selective separability properties.

In analogy to the (τ_i, τ_j) -tightness in [14] (τ_λ, τ_μ) -tightness was introduced by replacing τ_i and τ_j with τ_λ and τ_μ topologies respectively.

The (τ_λ, τ_μ) -tightness, $\lambda, \mu \in \Psi$, of a bisppace $(C(X), \tau_\lambda, \tau_\mu)$ is the least infinite cardinal κ such that whenever $A \subseteq C(X)$ and $f \in Cl_{\tau_\lambda}(A)$, there is $B \subseteq A$ such that $|B| \leq \kappa$ and $f \in Cl_{\tau_\mu}(B)$.

We recall that a subset A of X is called co-zero (or a functional open) set if $X \setminus A$ is a zero set. We mean by a zero set, a subset of X that is complete preimage of zero for certain function from $C(X)$.

Definition 3.1. A co-zero (functional open) family \mathbb{U} of X is called a λ - f -cover if X is not a member of \mathbb{U} and for each $A \in \lambda$ there is a $U \in \mathbb{U}$ such that $A \subseteq U$.

Note that λ - f -cover is the cover of $\bigcup \lambda$, but it can not be the cover of X . The symbol $\Lambda(\lambda)$ denotes the collection of all λ - f -covers for the family λ .

Definition 3.2. A space X is λ - μ -Lindelöf if for each $\mathcal{U} \in \Lambda(\lambda)$ there is a $\mathcal{V} \subseteq \mathcal{U}$ such that \mathcal{V} is countable and $\mathcal{V} \in \Lambda(\mu)$.

If $\lambda = \mu$ then we shall write it simply λ -Lindelöf. Note that if $\lambda = k$ then λ -Lindelöf is k -Lindelöf.

Theorem 3.3. For a space X , $\lambda, \mu \in \Psi$ and $\mu \subseteq \lambda$, the bispace $(C(X), \tau_\lambda, \tau_\mu)$ has countable (τ_λ, τ_μ) -tightness if and only if X is λ - μ -Lindelöf.

Proof. (\Rightarrow). Let $(C(X), \tau_\lambda, \tau_\mu)$ be countable (τ_λ, τ_μ) -tightness and $\mathcal{U} \in \Lambda(\lambda)$. For each pair of a element K of λ and $U \in \mathcal{U}$, $K \subseteq U$ let $f_{K,U}$ be any continuous function from X to $[0, 1]$ such that $f_{K,U}(K) \subseteq \{0\}$ and $f_{K,U}(X \setminus U) \subseteq \{1\}$. Let $A = \{f_{K,U} : K \in \lambda, K \subseteq U \in \mathcal{U}\}$. Then $\mathbf{0}$ belongs to the closure of A with respect to the τ_λ topology. Since $(C(X), \tau_\lambda, \tau_\mu)$ has countable (τ_λ, τ_μ) -tightness there is a countable set $B = \{f_{K_n, U_n} : n \in \mathbb{N}\}$ such that $\mathbf{0}$ belongs to the closure of B with respect to the τ_μ topology. We claim that $\{U_n : n \in \mathbb{N}\} \in \Lambda(\mu)$. Let $F \in \mu$. From the fact that $\mathbf{0}$ belongs to the closure of B with respect to the τ_μ topology it follows that there is an $i \in \mathbb{N}$ such that $[F, (-1, 1)]$ contains the function f_{K_i, U_i} . Then $F \subseteq U_i$. Otherwise for some $x \in F$ one has $x \notin U_i$ so that $f_{K_i, U_i}(x) = 1$, contradicting $f_{K_i, U_i} \in [F, (-1, 1)]$.

(\Leftarrow). Let A be a set of $C(X) \setminus \{\mathbf{0}\}$ the closure of which contain $\mathbf{0}$, with respect to the τ_λ topology. If $\{X\} \in \lambda$ (X is pseudocompact) then the τ_λ topology coincides with the C -compact-open topology, so $C_\lambda(X)$ is metrizable (Theorem 2.2 in [21]), thus first countable, which means that we can find a sequence $(a_n : n \in \mathbb{N})$, converging uniformly to $\mathbf{0}$ so there is nothing to be proved.

Let $\{X\} \notin \lambda$. For each $n \in \mathbb{N}$ and every set $K \in \lambda$ the neighborhood $[K, (-1/n, 1/n)]$ of $\mathbf{0}$ intersects A , so there exists a continuous function $f_{K,n} \in A$ such that $|f_{K,n}(x)| < 1/n$ for each $x \in K$. Since $f_{K,n}$ is a continuous function there is a co-zero set $U_{K,n}$ such that $f_{K,n}(U_{K,n}) \subseteq (-1/n, 1/n)$. Let $\mathcal{U}_n = \{U_{K,n} : K \in \lambda\}$.

As for any subset $K \in \lambda$ we have $K \neq X$, it can easily be achieved that none of the sets $U_{K,n}$ above coincides with X . So for each n , $\mathcal{U}_n \in \Lambda(\lambda)$. Each \mathcal{U}_n has countable μ - f -cover $\mathcal{V}_n \subseteq \mathcal{U}_n$. Define $B = \{f_{K,n} : n \in \mathbb{N}, U_{K,n} \in \mathcal{V}_n\}$. It is evident that $B \subseteq A$, $|B| \leq \aleph_0$, and $\mathbf{0}$ is the closure of B with respect to the τ_μ topology. Therefore the bispace $(C(X), \tau_\lambda, \tau_\mu)$ has countable (τ_λ, τ_μ) -tightness. \square

Corollary 3.4. The space $C_\lambda(X)$ has countable tightness if and only if X is λ -Lindelöf.

Definition 3.5. A space X has countable (τ_λ, τ_μ) -fan tightness if for each $x \in X$ and each sequence $(A_n : n \in \mathbb{N})$ of elements of $(\Omega_x)^\lambda$ there exists a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that, for each n , $B_n \subseteq A_n$ and $x \in cl_{\tau_\mu}(\bigcup_{n \in \mathbb{N}} B_n)$, i.e. if $S_{fin}((\Omega_x)^\lambda, (\Omega_x)^\mu)$ holds for each $x \in X$.

Definition 3.6. A space X has countable (τ_λ, τ_μ) -strong fan tightness if for each $x \in X$ $S_1((\Omega_x)^\lambda, (\Omega_x)^\mu)$ holds.

The proof of the next theorem follows much the similar lines as in the proof of the theorem 2.4 in [22].

Theorem 3.7. Let X be a Tychonoff space, $\lambda, \mu \in \Psi$ and $\mu \subseteq \lambda$. Then the following are equivalent:

1. $C(X)$ satisfies $S_1((\Omega_0)^\lambda, (\Omega_0)^\mu)$;
2. X has property $S_1(\Lambda(\lambda), \Lambda(\mu))$.

Proof. (1) \Rightarrow (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ -f covers of X . For each pair $A \in \lambda$ and a co-zero set U such that $A \subseteq U$ let $f_{A,U} : X \rightarrow [0, 1]$ be a continuous function with $f_{A,U}(A) \subseteq \{0\}$ and $f_{A,U}(X \setminus U) \subseteq \{1\}$.

Now let $B_n = \{f_{A,U} : A \in \lambda, A \subseteq U \in \mathcal{U}_n\}$. Now we claim that $\mathbf{0}$ is in the closure of each B_n , with respect to λ -open topology.

Indeed at first we have $\mathbf{0} \notin B_n$ and secondly let $\mathbf{0} \in [A, V]$ where $A \in \lambda$ and V is open subset of \mathbb{R} . There exists $U \in \mathcal{U}_n$ with $A \subseteq U$. For the functions $f_{A,U} \in B_n$ $f_{A,U} \in [A, V]$ and $f_{A,U}(A) = \{0\} \subseteq V$ hence $\mathbf{0}$ is in the closure of B_n with respect to λ -open topology, which means $B_n \in (\Omega_0)^\lambda$.

Now since $C(X)$ has countable (τ_λ, τ_μ) strong fan tightness there is a sequence $(f_{A_n, U_n} : n \in \mathbb{N})$ $\forall n \in \mathbb{N}$ with $A_n \in \lambda$, $U_n \in \mathcal{U}_n$ such that $\mathbf{0}$ belongs to the closure of $\{f_{A_n, U_n} : n \in \mathbb{N}\}$ with respect to the μ -open topology. To conclude the first part of the proof now we claim that $\{U_n : n \in \mathbb{N}\} \in \Lambda(\mu)$. Let $K \in \mu$. There is an $j \in \mathbb{N}$ such that $[K, (-1, 1)]$ contains the function f_{A_j, U_j} . Clearly $K \subseteq U_j$.

(2) \Rightarrow (1). Let $(B_m : m \in \mathbb{N}) \in \Omega_0^\lambda$. If $\{X\} \in \lambda$ it is clear. Now let $\{X\} \notin \lambda$. We set $B_{n,m} := B_{j(n,m)}$ for the bijection $j : \mathbb{N}^2 \mapsto \mathbb{N}$. For each $A \in \lambda$ the neighborhood $[A, (-1/n, 1/n)]$ of $\mathbf{0}$ intersects $B_{m,n}$, which means that there exists a continuous function $f_{A,m,n} \in B_{m,n}$ such that $|f_{A,m,n}(x)| < 1/n$ for each $x \in A$ and $f_{A,m,n}(U_{A,m,n}) \subseteq (-1/n, 1/n)$ for the co-zero set $U_{A,m,n}$.

Now we set $\mathcal{U}_{m,n} = U_{A,m,n}$, $A \in \lambda$. For any set $A \in \lambda$, $A \neq X$, none of the sets $U_{A,m,n}$ equals to X gives us. So for each m and n , $\mathcal{U}_{m,n} \in \Lambda(\lambda)$. To each sequence $(\mathcal{U}_{m,n} : m \in \mathbb{N})$ apply the fact that X is an $S_1(\Lambda(\lambda), \Lambda(\mu))$ -space. We can easily obtain a sequence $(U_{A_m,n} : m \in \mathbb{N})$, such that $\{U_{A_m,n} : m \in \mathbb{N}\} \in \Lambda(\mu)$. Now define $D = \{f_{A_m,n} : n \in \mathbb{N}, U_{A_m,n} \in \{U_{A_m,n} : m \in \mathbb{N}\}\}$. It is evident that $D \subseteq B_n$, $|D| \leq \aleph_0$, and $\mathbf{0}$ is the closure of D with respect to the τ_μ topology. Therefore the bispace $(C(X), \tau_\lambda, \tau_\mu)$ satisfies $S_1((\Omega_0)^\lambda, (\Omega_0)^\mu)$. \square

Corollary 3.8. For a Tychonoff space X the following are equivalent:

1. $C_\lambda(X)$ has countable strong fan tightness;
2. X has property $S_1(\Lambda(\lambda), \Lambda(\lambda))$.

In a similar way one can show that

Theorem 3.9. Let X be a Tychonoff space, $\lambda, \mu \in \Psi$ and $\mu \subseteq \lambda$. Then the following are equivalent:

1. $C(X)$ satisfies $S_{fin}((\Omega_0)^\lambda, (\Omega_0)^\mu)$;
2. X has property $S_{fin}(\Lambda(\lambda), \Lambda(\mu))$.

Corollary 3.10. The space $C_\lambda(X)$ has countable fan tightness if and only if X has property $S_{fin}(\Lambda(\lambda), \Lambda(\lambda))$.

The T -tightness $T(X)$ of a space X is the smallest infinite cardinal τ such that if $\{F_\alpha : \alpha < \kappa\}$ is an increasing family of closed subsets and $cf(\kappa) > \tau$, then $\bigcup\{F_\alpha : \alpha < \kappa\}$ is closed in X . This definition was introduced in [18] by Juhász. Since the family $\{F_\alpha : \alpha < \kappa\}$ is increasing and $cf(\kappa)$ is regular, we may say that the T -tightness $T(X)$ is the smallest infinite cardinal τ such that if $\{F_\alpha : \alpha < \kappa\}$ is an increasing family of closed subsets and κ is a regular cardinal greater than τ , then $\bigcup\{F_\alpha : \alpha < \kappa\}$ is closed in X . In [25] the T -tightness of function spaces $C_p(X)$ was investigated and in [12] it was considered for $C_k(X)$.

The bitopological version of this notion was introduced in [15].

A bispace $(X, \tau_\mu, \tau_\lambda)$ has countable (τ_μ, τ_λ) - T -tightness, if for each uncountable regular cardinal κ and each increasing sequence $(A_\alpha : \alpha < \kappa)$ of closed subsets of (X, τ_μ) the set $\bigcup\{A_\alpha : \alpha < \kappa\}$ is closed in (X, τ_λ) .

Theorem 3.11. Let X be a Tychonoff space, $\lambda, \mu \in \Psi$ and $\mu \subseteq \lambda$. Then (1) implies (2):

1. $(C(X), \tau_\mu, \tau_\lambda)$ has countable (τ_λ, τ_μ) - T - tightness;
2. for each regular cardinal κ and each increasing sequence $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of family of cozero subsets of X such that $\bigcup_{\alpha < \kappa} \mathcal{U}_\alpha$ is a λ - f -cover of X there is a $\beta < \kappa$ so that \mathcal{U}_β is a μ - f -cover of X .

Proof. (1) \Rightarrow (2). Since a bispaces $(C(X), \tau_\mu, \tau_\lambda)$ has countable (τ_λ, τ_μ) - T -tightness for each uncountable regular cardinal κ and each increasing sequence $\{A_\alpha : \alpha < \kappa\}$ of closed subsets of $(C(X), \tau_\lambda)$ the set $\bigcup \{A_\alpha : \alpha < \kappa\}$ is closed in $(C(X), \tau_\mu)$. Let for regular cardinal κ and increasing sequence $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of family of cozero subsets of X the family $\bigcup_{\alpha < \kappa} \mathcal{U}_\alpha$ be a λ - f -cover of X . Now we set $\mathcal{U}_{\alpha, K} := \{\mathcal{U} \in \mathcal{U}_\alpha : K \subset \mathcal{U}\}$ for each $\alpha < \kappa$ and $K \in \lambda$. For each $\mathcal{U} \in \mathcal{U}_{\alpha, K}$ let $f_{K, \mathcal{U}}$ be a continuous function from X into $[0, 1]$ such that $f_{K, \mathcal{U}}(K) = \{0\}$ and $f_{K, \mathcal{U}}(X \setminus \mathcal{U}) = \{1\}$. Now consider the set $A_\alpha = \{f_{K, \mathcal{U}} : \mathcal{U} \in \mathcal{U}_{\alpha, K}\}$, $\alpha < \kappa$.

By (1) we observe that the set $A = \bigcup_{\alpha < \kappa} cl_{\tau_\lambda}(A_\alpha)$ is closed in $(C(X), \tau_\mu)$.

Let $\langle \mathbf{0}, K, \varepsilon \rangle := [K, (-\varepsilon, \varepsilon)]$ be a standard basic τ_λ -neighborhood of $\mathbf{0}$. There exist $\alpha < \kappa$ and $\mathcal{U} \in \mathcal{U}_\alpha$ with $K \subset \mathcal{U}$. Then $\mathcal{U} \in \mathcal{U}_{\alpha, K}$, hence by construction there is $f \in A_\alpha \cap \langle \mathbf{0}, K, \varepsilon \rangle$. Therefore, each τ_λ -neighborhood of $\mathbf{0}$ intersects some A_α , $\alpha < \kappa$, i.e. $\mathbf{0}$ belongs to the τ_λ -closure of the set $\bigcup_{\alpha < \kappa} A_\alpha$ which is actually the set A . It follows that there is $\beta < \kappa$ with $\mathbf{0} \in cl_{\tau_\mu}(A_\beta)$. We claim that the corresponding family \mathcal{U}_β is an μ - f -cover of X .

Let $F \in \mu$. Then the τ_μ -neighborhood $\langle \mathbf{0}, F, 1 \rangle$ of $\mathbf{0}$ intersects A_β ; let $f_{F, \mathcal{U}} \in A_\beta \cap \langle \mathbf{0}, F, 1 \rangle$. Then $f_{F, \mathcal{U}}(X \setminus \mathcal{U}) = 1$ and thus $F \subset \mathcal{U} \in \mathcal{U}_\beta$. Then \mathcal{U}_β is an μ - f -cover of X . □

The following example show that the condition (2) may not involve the condition (1).

Example 1. Let $X = \omega_1 + 1$ be the space $\{\alpha : \alpha \leq \omega_1\}$ with the order topology, $\mu = p$, $\lambda = k$. Consider bitopological space $(C(X), \tau_p, \tau_k)$. Note that $C_p(X)$ has countable T -tightness (Theorem 2.3 in [25]) and hence for each regular cardinal κ and each increasing sequence $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of family of open subsets of X such that $\bigcup_{\alpha < \kappa} \mathcal{U}_\alpha$ is a ω -cover of X there is a $\beta < \kappa$ so that \mathcal{U}_β is a ω -cover of X . Thus $(C(X), \tau_p, \tau_k)$ has property that for each regular cardinal κ and each increasing sequence $\{\mathcal{U}_\alpha : \alpha < \rho\}$ of family of cozero subsets of X such that $\bigcup_{\alpha < \kappa} \mathcal{U}_\alpha$ is a k -cover of X there is a $\beta < \kappa$ so that \mathcal{U}_β is a ω -cover of X . Consider a set

$F_\beta = C(X) \setminus \{h \in C(X) : |h(x)| < \frac{1}{2} \text{ for } x \leq \beta \text{ and } |h(x)| < 1 \text{ for } x > \beta\}$ where β is a limit ordinal in X . Note that F_β is closed set in $C_k(X)$, but it is not closed set in $C_p(X)$. So $F = \bigcup_{\beta \leq \omega_1} F_\beta = F_{\omega_1}$. For $\mathbf{0}$ we have $\mathbf{0} \in cl_{\tau_p} F \setminus F$. It follows that bitopological space $(C(X), \tau_p, \tau_k)$ has not countable (τ_k, τ_p) - T -tightness.

Theorem 3.12. *Let X be a Tychonoff space, $\lambda \in \Psi$. Then the following are equivalent:*

1. $C_\lambda(X)$ has countable T -tightness;
2. for each regular cardinal ρ and each increasing sequence $(\mathcal{U}_\alpha : \alpha < \rho)$ of family of cozero subsets of X such that $\bigcup_{\alpha < \rho} \mathcal{U}_\alpha$ is a λ - f -cover of X there is a $\beta < \rho$ so that \mathcal{U}_β is a λ - f -cover of X .

Proof. (1) \Rightarrow (2). This follows from Theorem 3.11.

(2) \Rightarrow (1). Conversely, suppose that for each regular cardinal ρ and each increasing sequence $(\mathcal{U}_\alpha : \alpha < \rho)$ of family of cozero subsets of X such that $\bigcup_{\alpha < \rho} \mathcal{U}_\alpha$ is a λ - f -cover of X there is a $\beta < \rho$ so that \mathcal{U}_β is a λ - f -cover of X . Let κ be uncountable regular cardinal and $(A_\alpha : \alpha < \kappa)$ be increasing sequence of closed subsets of $C_\lambda(X)$.

Suppose $g \in \overline{\bigcup_{\alpha < \kappa} A_\alpha}$. Since $C_\lambda(X)$ is homogeneous, we may assume $g = f_0$ where $f_0 := \mathbf{0}$. For every $n \in \mathbb{N}$ and $\alpha < \kappa$, we set

$\mathcal{U}_{n,\alpha} = \{f^{-1}(W_n) : f \in A_\alpha\}$, where $W_n = (-\frac{1}{n}, \frac{1}{n})$, and $\mathcal{U}_n = \bigcup_{\alpha < \kappa} \mathcal{U}_{n,\alpha}$. Every \mathcal{U}_n is an λ -cover of X .

Indeed, let $F \in \lambda$ and consider the neighborhood $[F, W_n]$ of f_0 . By $f_0 \in \overline{\bigcup_{\alpha < \kappa} A_\alpha}$, there exist $\alpha < \kappa$ and $f \in A_\alpha \cap [F, W_n]$. Then $F \subset f^{-1}(W_n) \in \mathcal{U}_{n,\alpha}$. Thus, every \mathcal{U}_n is an λ -cover of X . For every $n \in \mathbb{N}$, we can find $\alpha_n < \kappa$ such that \mathcal{U}_{n,α_n} is an λ -cover of X . Let γ be the supremum of α_n 's. Then for every $n \in \mathbb{N}$, $\mathcal{U}_{n,\gamma}$ is an λ -cover of X . Now we claim $f_0 \in A_\gamma$. Let $[F, W]$ be any neighborhood of f_0 and choose $n \in \mathbb{N}$ with $W_n \subset W$. Since $\mathcal{U}_{n,\gamma}$ is an λ -cover of X , there exists an $f \in A_\gamma$ such that $F \subset f^{-1}(W_n)$. Then $f \in A_\gamma \cap [F, W]$. Thus $f_0 \in \overline{A_\gamma} = A_\gamma$. We conclude that $\bigcup_{\alpha < \kappa} A_\alpha$ is closed in $C_\lambda(X)$. □

4. Bitopological \mathcal{R} -separability and \mathcal{M} -separability

\mathcal{R} -separability and \mathcal{M} -separability in bitopological spaces were first introduced and studied in [14]. We have some analogous results on \mathcal{R} -separability

and \mathcal{M} -separability of bitopological functional spaces endowed with set open topology.

Definition 4.1. Let X be a topological space and λ be a family subsets of X . Then the space X said to be separably λ -submetrizable if there are separable metric space Y , continuous map f the space X onto Y such that f is one-to-one on $\bigcup \lambda$.

Note that if $\bigcup \lambda = X$ then X is a submetrizable space.

For example we consider the Mrowka-Isbell space. Let \mathcal{M} be a maximal infinite family of infinite subsets of \mathbb{N} such that the intersection of any two members of \mathcal{M} is finite, and let $\Psi = \mathbb{N} \bigcup \mathcal{M}$, where a subset U of Ψ is defined to be open provided that for any set $M \in \mathcal{M}$, if $M \in U$ then there is a finite subset F of M such that $\{M\} \bigcup M \setminus F \subset U$.

So the Mrowka-Isbell space is separably λ -submetrizable space (where λ is a family finite subsets of set \mathbb{N} of isolate points of space Ψ), but it is not submetrizable space.

Theorem 4.2. *Let X be a Tychonoff space, $\lambda \in \Psi$. Then following conditions are equivalent:*

1. $C_\lambda(X)$ is a separable space;
2. X is a separably λ -submetrizable space.

Proof. (1) \Rightarrow (2). Let D be a countable dense subset of $C(X)$. Let us observe that for the diagonal map $f = \Delta_{f_i \in D} f_i$ we have $f : X \mapsto Y$ where $Y = f(X) \subseteq \prod_{i \in \mathbb{N}} \mathbb{R}_i$ and f is one-to-one on $\bigcup \lambda$.

(2) \Rightarrow (1). Consider the set $C(f(X))$ with $f(\lambda)$ -open topology. Note that $f(\lambda)$ is the family of compact subsets of $f(X)$ and it is closed under compact subsets $f(X)$ of its elements. By Theorem (N.Noble, [17]), the space $C_c(f(X))$ is separable space then $C_{f(\lambda)}(f(X))$ is a separable space. It follows immediately that $C_\lambda(X)$ is a separable space. □

Corollary 4.3. Let X be a Tychonoff space, $\lambda \in \Psi$ and let $C_\lambda(X)$ be a separable space. Then any element of family λ is a metrizable compact subset of X .

Theorem 4.4. *Let X be a Tychonoff separably λ -submetrizable space, $\lambda, \mu \in \Psi$ and $\mu \subseteq \lambda$. Then the following are equivalent:*

1. $X \in S_1(\Lambda(\lambda), \Lambda(\mu));$
2. $(C(X), \tau_\mu, \tau_\lambda)$ is $R_{(\tau_\lambda, \tau_\mu)}$ - separable.

Proof. (1) \Rightarrow (2). By Theorem 4.2, $C_\lambda(X)$ is separable. On the other hand, by Theorem 3.7, the bispaces $(C(X), \tau_\lambda, \tau_\mu)$ has countable (τ_λ, τ_μ) - strong fan tightness if (and only if) $X \in S_1(\Lambda(\lambda), \Lambda(\mu))$. By Corollary 3 in [14], we obtain $(C(X), \tau_\mu, \tau_\lambda)$ is $R_{(\tau_\lambda, \tau_\mu)}$ - separable.

(2) \Rightarrow (1). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ -covers of X . For every $n \in \mathbb{N}$ let $A_n = \{f \in C_\lambda(X) : \text{there is } U \in \mathcal{U}_n, f(X \setminus U) = \{1\}\}$.

First, we verify that each A_n is dense in $C_\lambda(X)$. To this end let us consider $f \in C_\lambda(X)$ and let $\bigcap_{i \leq m} [K_i, V_i]$ be a basic neighbourhood of f . The set $K = \bigcup_{i \leq m} K_i$ is compact and $K \in \lambda$, and since \mathcal{U}_n is a λ -cover, there is $U \in \mathcal{U}_n$ containing K . There is also $g \in C_\lambda(X)$ such that $g(X \setminus U) = \{1\}$ and $g \upharpoonright K = f \upharpoonright K$. Then $g \in \bigcap_{i \leq m} [K_i, V_i] \cap A_n$.

Since $(C(X), \tau_\mu, \tau_\lambda)$ is $R_{(\tau_\lambda, \tau_\mu)}$ - separable there are functions $f_n \in A_n$, $n \in \mathbb{N}$, such that the set $\{f_n : n \in \mathbb{N}\}$ is dense in $C_\mu(X)$. Let $U_n \in \mathcal{U}_n$ be a set for which $f_n(X \setminus U_n) = \{1\}$ holds. We claim that $\{U_n : n \in \mathbb{N}\} \in \Lambda(\mu)$. Let $F \in \mu$. Suppose that there is a point $x_n \in F \setminus U_n$ for each $n \in \mathbb{N}$ which means $f_n(x_n) = 1$ and it contradicts the fact that $\{f_n : n \in \mathbb{N}\}$ is dense in $C_\mu(X)$. □

Corollary 4.5. Let X be a Tychonoff separably λ -submetrizable space, $\lambda \in \Psi$. Then the following are equivalent:

1. $X \in S_1(\Lambda(\lambda), \Lambda(\lambda));$
2. $C_\lambda(X)$ is R - separable.

Theorem 4.6. Let X be a Tychonoff separably λ -submetrizable space, $\lambda, \mu \in \Psi$ and $\mu \subseteq \lambda$. Then the following are equivalent:

1. $X \in S_{fin}(\Lambda(\lambda), \Lambda(\mu));$
2. $(C(X), \tau_\mu, \tau_\lambda)$ is $M_{(\tau_\lambda, \tau_\mu)}$ - separable.

Proof. (1) \Rightarrow (2). The space $C_\lambda(X)$ is separable since X is separably λ -submetrizable space. On the other hand, by Theorem 3.11, (τ_λ, τ_μ) -fan tightness of $(C(X), \tau_\mu, \tau_\lambda)$ is countable if and only if X has selection property $S_{fin}(\Lambda(\lambda), \Lambda(\mu))$ and now apply Corollary 6 in [14].

(2) \Rightarrow (1). Assume that $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ -covers. For every $n \in \mathbb{N}$ we set

$V_n = \{f \in C_\lambda(X) : \text{there is } U \in \mathcal{U}, f(X \setminus U) = \{1\}\}.$

We follow the proof of Theorem 4.4 to show that each V_n is dense in $C_\lambda(X)$.

By the hypothesis $(C(X), \tau_\mu, \tau_\lambda)$ is $M_{(\tau_\lambda, \tau_\mu)}$ - separable there are finite sets $W_n = \{f_{n,1}, \dots, f_{n,m_n}\} \subset V_n$, $n \in \mathbb{N}$, such that the set $\bigcup_{n \in \mathbb{N}} W_n$ is dense in $C_\mu(X)$.

Now consider the set $\mathbb{D}_n = \{U_{n,1}, \dots, U_{n,m_n} : f_{n,i}(X \setminus U_{n,i}) = \{1\}, i \leq m_n\}$ which is a finite subset of \mathcal{U}_n . It remains to show that $\bigcup_{n \in \mathbb{N}} \mathbb{D}_n \in \Lambda(\mu)$. Let $F \in \mu$. For some $j \in \mathbb{N}$ we have $[F, (-1, 1)] \cap W_j$, i.e. there is a function $f_{j,m_j} \in W_j$ such that $f_{j,m_j}(x) \in (-1, 1)$ for each $x \in F$. This means $F \subset U_{j,m_j}$ as required. □

Corollary 4.7. Let X be a Tychonoff separably λ -submetrizable space, $\lambda \in \Psi$. Then the following are equivalent:

1. $X \in S_{fin}(\Lambda(\lambda), \Lambda(\lambda))$;
2. $C_\lambda(X)$ is M - separable.

Recall that a bispace (X, τ_1, τ_2) is (τ_i, τ_j) -Pytkeev ($i \neq j; i, j = 1, 2$) [13] (see also [22]) if for each $A \subset X$ and each $x \in Cl_i(A) \setminus A$ there are infinite sets $B_n \subset A$, $n \in \mathbb{N}$, such that each τ_j -neighbourhood of x contains some B_n .

By Theorem 9 in [14], we have following result.

Theorem 4.8. Let X be a Tychonoff separably λ -submetrizable space, $\lambda, \mu \in \Psi$ and $\mu \subseteq \lambda$. If $(C(X), \tau_\mu, \tau_\lambda)$ is $M_{(\tau_\mu, \tau_\lambda)}$ - separable and (τ_μ, τ_λ) -Pytkeev bispace, then it is $R_{(\tau_\mu, \tau_\lambda)}$ - separable.

Corollary 4.9. Let X be a Tychonoff separably λ -submetrizable space, $\lambda \in \Psi$. If $C_\lambda(X)$ is M - separable and Pytkeev space, then it is R - separable.

5. Bitopological H -separability and GN -separability

In this section we will be interested in some results on bitopological H -separability and GN -separability.

We begin by recalling the notion of weak selectively Reznichenko property for the bitopological spaces. A bispace (X, τ_1, τ_2) has the weak selectively (τ_i, τ_j) -Reznichenko property ($i \neq j; i, j = 1, 2$), if for each sequence $(A_n : n \in \mathbb{N})$ of subsets of X and each point $x \in \bigcap_{n \in \mathbb{N}} Cl_i(A_n)$ there are finite sets

$B_n \subset A_n$, $n \in \mathbb{N}$, such that each τ_j -neighbourhood of x intersects B_n for all but finitely many n .

The definition of the selective bitopological version of the Reznichenko property was given in [22]. It has been characterized by considering the compact-open and the topology of pointwise convergence on the set of all continuous real-valued functions in [23].

The notion of weak selectively Reznichenko property was introduced in [14]. The following results may be proved in much the same way as Theorem 10 and Theorem 11 in [14].

Theorem 5.1. *Let X be a Tychonoff separably λ -submetrizable space, $\lambda, \mu \in \Psi$ and $\mu \subseteq \lambda$. Then the following are equivalent:*

1. $(C(X), \tau_\mu, \tau_\lambda)$ is $H_{(\tau_\lambda, \tau_\mu)}$ - separable;
2. For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of λ -covers there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each n , $\mathcal{V}_n \subset \mathcal{U}_n$ and each $F \in \mu$ is contained in an element of \mathcal{V}_n for all but finitely many $n \in \mathbb{N}$.

Proof. (1) \Rightarrow (2). Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ -covers. For every $n \in \mathbb{N}$ let

$$D_n = \{f \in C_\lambda(X) : \text{there is } U \in \mathcal{U}_n, f(X \setminus U) = \{1\}\}.$$

Easily one can prove that each D_n is dense in $(C(X), \tau_\lambda)$. Since $(C(X), \tau_\mu, \tau_\lambda)$ is $H_{(\tau_\lambda, \tau_\mu)}$ - separable there are finite sets $F_n \subset D_n$, $n \in \mathbb{N}$, such that each τ_μ -open set intersects F_n for all but finitely many n . Let \mathcal{V}_n , $n \in \mathbb{N}$, be the family of sets $U_f \in \mathcal{U}_n$, $f \in F_n$, such that $f(X \setminus U_f) = \{1\}$. We need to verify that the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ witnesses that X satisfies (2).

Let $K \in \mu$. The open neighborhood $H = [K, (-1, 1)]$ intersects F_m for each m bigger than m_0 ; now pick $f_m \in H \cap F_m$, $m > m_0$. Then for each $m > m_0$, $K \subset U_{f_m} \in \mathcal{V}_m$, as required in (2).

(2) \Rightarrow (1). The proof consists of two parts.

Claim 1. $C(X)$ has the weak selectively (τ_λ, τ_μ) -Reznichenko property.

We take a sequence $(D_n : n \in \mathbb{N})$ of subsets of $C(X)$ with the τ_λ -closures of which contain $\mathbf{0}$. For every $T \in \lambda$ and every $m \in \mathbb{N}$ the τ_λ -neighborhood $[T, \frac{1}{m}]$ of $\mathbf{0}$ intersects each D_n . So for each $n \in \mathbb{N}$ there exists a function $f_{T,n,m} \in D_n$ satisfying $|f_{T,n,m}(x)| < \frac{1}{m}$ for each $x \in T$. For each n set

$$\mathcal{U}_{n,m} = \{f^{-1}(-\frac{1}{m}, \frac{1}{m}) : m \in \mathbb{N}, f \in D_n\}.$$

(For a bijection $\varphi : \mathbb{N}^2 \mapsto \mathbb{N}$ we put $\mathcal{U}_{n,m} := \mathcal{U}_{\varphi(m,n)}$). We claim that for each $n, m \in \mathbb{N}$, each $C \in \lambda$ is contained in an element of $\mathcal{U}_{n,m}$. Indeed, if

$C \in \lambda$, then there is $f_{C,n,m} \in [C, \frac{1}{m}] \cap D_n$. Hence $|f_{C,n,m}(x)| < \frac{1}{m}$ for each $x \in C$. This shows that $C \subset f_{C,n,m}^{-1}(-\frac{1}{m}, \frac{1}{m}) \in \mathcal{U}_{n,m}$.

Put $S := \{m \in \mathbb{N} : X \in \mathcal{U}_{n,m} \text{ for some } n \in \mathbb{N}\}$. There are two cases to consider.

Case 1. S is infinite.

There are $m_1 < m_2 < \dots$ in M and (the corresponding) n_1, n_2, \dots in \mathbb{N} such that $f_{T_i, n_i, m_i}^{-1}(-\frac{1}{m_i}, \frac{1}{m_i}) = X$ for all $i \in \mathbb{N}$ and some $T_i \in \lambda$. Let $[R, \epsilon]$ be a τ_μ -neighborhood of $\mathbf{0}$. Pick m_k such that $\frac{1}{m_k} < \epsilon$. For every $m_i > m_k$ we have $f_{T_i, n_i, m_i}(x) \in (-\frac{1}{m_i}, \frac{1}{m_i})$ for each $x \in X$ and so $f_{T_i, n_i, m_i} \in [R, \frac{1}{m_i}] \subset [R, \epsilon]$. This means that the sequence $(f_{T_i, n_i, m_i} : i \in \mathbb{N})$ τ_μ -converges to $\mathbf{0}$, hence $C(X)$ has the weak selectively (τ_λ, τ_μ) -Reznichenko property at $\mathbf{0}$.

Case 2. Consider the case S is finite.

There is $m_0 \in \mathbb{N}$ such that for each $m \geq m_0$ and each $n \in \mathbb{N}$, the set $\mathcal{U}_{n,m}$ is a λ -cover of X . We may suppose $m_0 = 1$. Further, we can consider only λ -covers $\mathcal{U}_{n,n}$, $n \in \mathbb{N}$. We can apply the condition (2) of this theorem to the sequence $\mathcal{U}_{n,n}$ to get a sequence $\mathcal{V}_{n,n}$ where for each $n \in \mathbb{N}$ $\mathcal{V}_{n,n}$ is a finite subset of $\mathcal{U}_{n,n}$ so that each $R \in \mu$ belongs to some $V \in \mathcal{V}_{n,n}$ for all but finitely many n . Choose the corresponding functions $f_{T_V, \frac{1}{n}, \frac{1}{n}}$, $V \in \mathcal{V}_{n,n}$, and put $F_n = \{f_{T_V, \frac{1}{n}, \frac{1}{n}} : V \in \mathcal{V}_{n,n}\}$. Then each F_n is a finite subset of D_n . Let $[R, \frac{1}{i}]$ be a neighborhood of $\mathbf{0}$. Let n_0 be such that $\frac{1}{n} < \frac{1}{i}$ and for each $n > n_0$ there is $V_n \in \mathcal{V}_{n,n}$ containing R . Choose a corresponding $f_n \in F_n$. Since this can be done for all $n > n_0$, we conclude that for all $n > n_0$ we have $f_n \in [R, \frac{1}{i}]$, i.e., $F_n \cap [R, \frac{1}{i}] \neq \emptyset$ for all $n > n_0$.

We now get back to proving the theorem.

Claim 2. $C(X)$ is $H(\tau_\lambda, \tau_\mu)$ -separable.

Since X is a Tychonoff separably λ -submetrizable space, and $\tau_\mu \leq \tau_\lambda$, there is a countable dense subset $D = \{d_n : n \in \mathbb{N}\}$ in $(C(X), \tau_\lambda)$ so also in $(C(X), \tau_\mu)$. Let $(E_n : n \in \mathbb{N})$ be a sequence of dense subsets of $(C(X), \tau_\lambda)$. Fix $m \in \mathbb{N}$. Since $d_m \in Cl_{\tau_\lambda}(E_n)$ for each $n \in \mathbb{N}$, and $C(X)$ has the weak selectively (τ_λ, τ_μ) -Reznichenko property, there are finite sets $R_{n,m}$, such that for each n , $R_{n,m} \subset E_n$ and each τ_μ -neighborhood of d_m intersects all but finitely many $R_{n,m}$. For each n put $R_n = \bigcup \{R_{n,m} : m \leq n\}$. The sequence $(R_n : n \in \mathbb{N})$ witnesses for $(E_n : n \in \mathbb{N})$ that $C(X)$ is $\mathcal{H}_{(\tau_\lambda, \tau_\mu)}$ -separable. Indeed, let G be an open set in $(C(X), \tau_\mu)$. Then there is $d_m \in G$, hence G meets all but finitely many R_n .

□

Corollary 5.2. Let X be a Tychonoff separably λ -submetrizable space, $\lambda \in \Psi$. Then the following are equivalent:

1. $C_\lambda(X)$ is H - separable;
2. For each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of λ -covers there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each n , $\mathcal{V}_n \subset \mathcal{U}_n$ and each $F \in \lambda$ is contained in an element of \mathcal{V}_n for all but finitely many $n \in \mathbb{N}$.

Theorem 5.3. *If $(C(X), \tau_\mu, \tau_\lambda)$ is $GN_{(\tau_\lambda, \tau_\mu)}$ - separable, then X satisfies $S_{fin}(\Lambda(\lambda), \Lambda(\mu)^{gp})$.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of λ -covers of X . Now we define sets D_n in $(C(X), \tau_\lambda)$ as in the proof of Theorem 5.1. These sets are d -dense in $C(X)$. Apply the fact that $C(X)$ is $\mathcal{GN}_{(\tau_\lambda, \tau_\mu)}$ -separable, there are $f_n \in D_n$, $n \in \mathbb{N}$, such that $D = \{f_n : n \in \mathbb{N}\}$ is τ_μ -groupable, i.e. $D = \bigcup_{m \in \mathbb{N}} G_m$, where each $G_m = \{f_m^{k_1}, \dots, f_m^{k_m}\}$ is a finite subset of D and each τ_μ -open set meets all but finitely many G_m . For each $m \in \mathbb{N}$, let

$$\mathcal{V}_m = \{U_m^{k_i} : f_m^{k_i}(X \setminus U_m^{k_i}) = \{1\}, i \leq m\}.$$

Let us show that each $F \in \mu$ is contained in some $V \in \mathcal{V}_m$ for all but finitely many m . Let $F \in \mu$. Then the τ_μ -open set $[F, 1]$ intersects G_m for all m bigger than some $m_0 \in \mathbb{N}$. Let $f_m^{k_j} \in [F, 1] \cap G_m$, $m \geq m_0$. Then $F \subseteq U_m^{k_j}$, $m \geq m_0$. It shows that X has $S_{fin}((\Lambda(\lambda), \Lambda(\mu)^{gp}))$. □

Corollary 5.4. If $C_\lambda(X)$ is GN - separable, then X satisfies $S_{fin}(\Lambda(\lambda), \Lambda(\lambda)^{gp})$.

From Theorem 3.7, Theorem 5.3 and Corollary 5.4 we obtain

Theorem 5.5. *If $(C(X), \tau_\mu, \tau_\lambda)$ is $GN_{(\tau_\lambda, \tau_\mu)}$ - separable, then X satisfies $S_1(\Lambda(\lambda), \Lambda(\mu))$ and $S_{fin}(\Lambda(\lambda), \Lambda(\mu)^{gp})$.*

Corollary 5.6. If $(C(X), \tau_\mu, \tau_\lambda)$ is $GN_{(\tau_\lambda, \tau_\mu)}$ - separable, then $(C(X), \tau_\mu, \tau_\lambda)$ is $R_{(\tau_\lambda, \tau_\mu)}$ - separable as well as $H_{(\tau_\lambda, \tau_\mu)}$ - separable.

Corollary 5.7. If $C_\lambda(X)$ is GN - separable, then $C_\lambda(X)$ is R - separable as well as H - separable.

6. Examples

We consider some examples which separating different type of separability in function bispaces.

Example 6.1. Let $\mathbb{I} = [0, 1] \subset \mathbb{R}$.

- By Example 2.14 in [4], $C_p(\mathbb{I})$ is M -separable i.e. \mathbb{I} has the property $S_{fin}(\Lambda(p), \Lambda(p))$ by Corollary 4.7.

Since each k -cover of \mathbb{I} is ω -cover we have that $\mathbb{I} \in S_{fin}(\Lambda(k), \Lambda(p))$. Hence the space $(C(\mathbb{I}), \tau_p, \tau_k)$ is $M_{(\tau_k, \tau_p)}$ - separable.

- By Proposition 61 in [3], $C_p(\mathbb{I})$ is not R -separable, and, by Fact 2.1 in [14], space $(C(\mathbb{I}), \tau_p, \tau_k)$ is not $R_{(\tau_k, \tau_p)}$ - separable.

It follows that the bitopological space $(C(\mathbb{I}), \tau_p, \tau_k)$ is $M_{(\tau_k, \tau_p)}$ - separable, but it is not $R_{(\tau_k, \tau_p)}$ - separable.

Example 6.2. Let $X = \omega^\omega$. Then X is p -Lindelöf space, but $C_p(X)$ is not M -separable.

Recall that \mathfrak{b} denote the minimum of cardinality of an unbounded set in ω^ω [7].

Example 6.3. Let X be an uncountable, second countable space of cardinality less than \mathfrak{b} . By Corollary 43 in [3], $C_p(X)$ is H -separable, but $C_p(X)$ is not R -separable, and, hence, $C_p(X)$ is not GN -separable.

Question 6.4. Does there exist an X such that $(C(X), \tau_\mu, \tau_\lambda)$ is $R_{(\tau_\lambda, \tau_\mu)}$ - separable and $H_{(\tau_\lambda, \tau_\mu)}$ - separable, but it isn't $GN_{(\tau_\lambda, \tau_\mu)}$ - separable for some $\lambda, \mu \in \Psi$ and $\mu \subseteq \lambda$ (for $\lambda = k$ and $\mu = p$)?

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